Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 73–84

Contact trivialization of ordinary differential equations[1]

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Abstract. We find the explicit conditions under which a single ordinary differential equation (shortly ODE) of order ≥ 4 is (locally) equivalent to the trivial equation under the group of contact transformations. The computation of these conditions is based on the canonical Cartan connection associated with any ordinary differential equation of higher order.

Keywords. Equivalence of ODE's, contact invariants, graded Lie algebra, Cartan connection.

MS classification. 34A26, 53B15.

1. Examples of trivialization problems

We say that an ODE is *trivializable* if it can be brought to the trivial equation

$$y^{(k)} = 0$$

by a certain contact transformation. The trivialization problem, that is the description of all trivializable equations, is a part of the more general equivalence problem of differential equations. The equivalence problem, in its turn, can be essentially reduced to computation of invariants of ODE's under the group of contact transformations. As we show in this paper, to find whether the given equation is trivializable or not we need to compute only a finite set of such invariants and check that they vanish on the equation. The major idea in computing these invariants is that any ordinary differential equation of order ≥ 3 can be considered as a certain finite-type geometric structure, and invariants of the equation are just the invariants of some coframe canonically associated with this structure.

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

Below we present several well-known examples of such invariants for other classes of differential equations.

Example 1. Consider the class of linear homogeneous differential equations

$$y^{(n+1)} + p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

viewed up to the transformations $(x, y) \mapsto (\lambda(x), \mu(x)y)$, which are the most general transformations preserving the class of linear homogeneous equations. The invariants of these equations were constructed by E. Wilczynski ([13]) and can be described as follows.

First, any linear equation can be always brought to the so-called Laguerre–Forsyth canonical form:

$$y^{(n+1)} + q_{n-2}(x)y^{(n-2)} + \cdots + q_0(x)y = 0.$$

The set of transformations preserving this canonical form is already a finite--dimensional Lie group:

$$(x, y) \mapsto \left(\frac{ax + b}{cx + d}, \frac{ey}{(cx + d)^{n+1}}\right).$$

This group acts on coefficients $q_0, \dots, q_{n-2}$ of the canonical form, and the invariants of this action are identified with the invariants of the general linear equation. There are precisely $n - 1$ such invariants $\theta_3, \dots, \theta_{n+1}$, where

$$\theta_k = \sum_{j=1}^{k-2} (-1)^j \frac{(2k - j - 1)!\,(n - k + j)!}{(k - j)!\,j!} q_{n-k+j}^{(j-1)}, \quad k = 3, \dots, n + 1. \tag{1}$$

Though these formulas express invariants in coefficients of the canonical form, they can be written explicitly in terms of the initial coefficients of the general equation. For example, the simplest of these invariants is:

$$\theta_3 = p_n'' + \frac{6}{n+1} p_n p_n' + \frac{4}{(n+1)^2} p_n^3 + \frac{6}{n} p_{n-1}' - \frac{12}{n(n+1)} p_n p_{n-1} + \frac{12}{n(n-1)} p_{n-2}.$$

Let us notice that all these invariants are polynomials in $p_0, \dots, p_n$, the coefficients of the initial equation and their derivatives. In the sequel we shall call the invariants $\theta_3, \dots, \theta_{n+1}$ *Wilczynski invariants of linear differential equations*.

One of the main results of Wilczynski says that an arbitrary linear equation is equivalent to the trivial one if and only if all these invariants vanish identically. This gives a simple and effective way to check whether a linear equation is trivializable or not.

In the next examples and throughout the paper we shall use the following notation. Let us fix an $(n + 1)$-th order ordinary differential equation

$$y^{(n+1)} = f(x, y, y', \dots, y^{(n)}). \tag{2}$$

Then for any function $F(x, y, \dots, y^{(n)})$ we denote by F_i the partial derivative

$\partial F/\partial y^{(i)}$, $i = 0, \dots, n$, and by F_x the total derivative

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \sum_{i=0}^{n-1} y^{(i+1)} \frac{\partial F}{\partial y^{(i)}} + f \frac{\partial F}{\partial y^{(n)}}.$$

Example 2. Consider the set of second-order ODE's $y'' = f(x, y, y')$ viewed up to the pseudogroup of all point transformations, that is, arbitrary (locally) invertible transformations of the form $(x, y) \mapsto (\lambda(x, y), \mu(x, y))$. The geometry lying behind this equation was explored by E. Cartan ([2]), who associated the the canonical coframe with this equation and proved that all its invariants can be derived by covariant differentiation from the following two:

$$I_1 = f_{1111} = \frac{\partial^4 f}{(\partial y')^4};$$

$$I_2 = \tfrac{1}{6} f_{11xx} - \tfrac{1}{6} f_1 f_{11x} - \tfrac{2}{3} f_{01x} + \tfrac{2}{3} f_1 f_{01} + f_{00} - \tfrac{1}{2} f_0 f_{11}.$$

In particular, the general equation is equivalent to the trivial one if and only if $I_1 = I_2 = 0$.

Let us notice that the local contact geometry of second-order ODE's is trivial, since any two such equations are locally equivalent under the pseudogroup of all contact transformations (see, for example, [8]).

Example 3. The contact geometry of third order ODE's $y''' = f(x, y, y', y'')$ was considered by S.-S. Chern ([3]) who also constructed a canonical coframe for these equations and found two similar invariants:

$$I_1 = f_{2222} = \frac{\partial^4 f}{(\partial y'')^4};$$

$$I_2 = -f_0 - \tfrac{1}{3} f_1 f_2 - \tfrac{2}{27} f_2^3 + \tfrac{1}{2} f_{1x} + \tfrac{1}{3} f_2 f_{2x} - \tfrac{1}{6} f_{2xx}.$$

Again, the general equation is trivializable if and only if these invariants vanish identically.

The goal of this paper is to extend the last example to the case of ODE's of order greater then 3. Our main results can be formulated as follows:

Theorem 1. *Let $\theta_i = F_i(p_j^{(k)})$ be Wilczynski invariants of linear $(n+1)$-th order ODE's for some polynomials F_i, $i = 3, \dots, n+1$. Then the expressions*

$$L_i = F_i\left(\left(\frac{d}{dx}\right)^k (-f_j)\right), \quad i = 3, \dots, n+1,$$

are contact invariants of general $(n+1)$-th order ordinary differential equations for any $n \geq 2$.

In particular, it is easy to check that for the third order ODE the invariant L_3 coincides up to a constant with the Chern invariant I_2.

Theorem 2. *The equation $y^{(n+1)} = f(x, y, \dots, y^{(n)})$ is trivializable if and only if the function f satisfies the following relations*: $L_3 = \dots = L_{n+1} = 0$ *and additionally*

for $n = 3$: $f_{333} = 6f_{233} + f_{33}^2 = 0$;
for $n = 4$: $f_{44} = 6f_{234} - 4f_{333} - 3f_{34}^2 = 0$;
for $n = 5$: $f_{55} = f_{45} = 0$;
for $n \geq 6$: $f_{n,n} = f_{n,n-1} = f_{n-1,n-1} = 0$.

The proofs of these theorems will be presented in the following sections.

2. Canonical Cartan connection

In this section we outline the construction of the canonical Cartan connection associated with any ordinary differential equation starting from order 4. The further details and references can be found in ([4]).

2.1. ODE as a filtered manifold

An arbitrary $(n+1)$-th order ordinary differential equation (2) can be considered as a hypersurface $\mathcal{E}$ in the space J^{n+1} of $(n+1)$-jets of mappings from $\mathbb{R}$ to $\mathbb{R}$ (or, more generally, of arbitrary one-dimensional submanifolds in the plane). Let $(x, y = y^0, y^1, \dots, y^{n+1})$ be the standard coordinate system on J^{n+1}. Then $\mathcal{E}$ is given by the equation $y^{n+1} = f(x, y^0, \dots, y^n)$, and the functions $(x, y^0, \dots, y^n)$ restricted to $\mathcal{E}$ give the coordinate system on $\mathcal{E}$. Let $\tau = -dx$, $\theta^i = dy^i - y^{i+1}\,dx$, $i = 0, \dots, n-1$, $\theta^n = dy^n - f\,dx$ be a fixed coframe on $\mathcal{E}$.

The surface $\mathcal{E}$ inherits the following geometric structures from the jet space J^{n+1}, invariant under contact transformations. First, there is a filtration

$$0 = T^0\mathcal{E} \subset T^{-1}\mathcal{E} \subset \dots \subset T^{-n-1}\mathcal{E} = T\mathcal{E},$$

where $T^{-i}\mathcal{E}$ is the inverse image of contact distribution on J^{n-i+1} under the projection $\pi_{n+1,n-i+1} : J^{n+1} \to J^{n-i+1}$ restricted to $\mathcal{E}$. Notice that $T^{-i}\mathcal{E}$ is the annihilator of the forms $\theta^0, \dots, \theta^{n-i}$ for any $i > 0$.

Then there is a one-dimensional distribution $E \subset T^{-1}\mathcal{E}$, whose integral curves are lifts of solutions of our equation. It is the annihilator of the forms $\theta^0, \dots, \theta^n$. Finally, there is a one-dimensional distribution $F \subset T^{-1}\mathcal{E}$, whose integral curves are fibers of the projection $\pi_{n+1,n-1}$ restricted to $\mathcal{E}$. It is equal to the characteristic distribution of $T^{-2}\mathcal{E}$, and is the annihilator of the forms $\tau, \theta^0, \dots, \theta^{n-1}$.

We call any coframe $\omega^0, \dots, \omega^n, \omega^x$ on $\mathcal{E}$ *adapted to the equation* (2), if

(a) the annihilator of forms $\omega^0, \dots, \omega^{n-i}$ is equal to $T^{-i}\mathcal{E}$ for all $i = 1, \dots, n$;
(b) the annihilator of forms $\omega^0, \dots, \omega^n$ is equal to E;
(c) the annihilator of $\omega^x, \omega^0, \dots, \omega^{n-1}$ is equal to F;
(d) $d\omega^i + \omega^x \wedge \omega^{i+1} = 0 \mod \langle \omega^0, \dots, \omega^i \rangle$ for all $i = 0, \dots, n-1$.

It is easy to check that any adapted coframe has the form:

$$\omega^x = A\tau + \sum_{j=0}^{n-1} A^x_j \theta^j, \quad \omega^i = (B/A^i)\theta^i + \sum_{j=0}^{i-1} A^i_j \theta^j,\ i = 0, \dots, n, \tag{3}$$

where the functions A and B do not vanish anywhere.

2.2. Adapted Cartan connections

Let $\mathbb{R}[x, y]$ denote the ring of real polynomials in x and y. Let $GL(2, \mathbb{R})$ act on $\mathbb{R}[x, y]$ in the usual way via linear substitutions in x and y. The infinitesimal version of this action is generated by the Lie algebra $\mathfrak{a}$ spanned by four vector fields

$$X = x\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}, \quad H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Let $V_n \subset \mathbb{R}[x, y]$ denote the subspace consisting of homogeneous polynomials of degree n. Then it is well known that V_n is an irreducible $GL(2, \mathbb{R})$-module. It will occasionally be necessary to refer to a basis of V_n. For this purpose, we use the basis given by the monomials

$$E_i = \frac{x^{n-i}y^i}{i!}, \quad i = 0, \dots, n.$$

Let G be the semidirect product of $GL(2, \mathbb{R})$ and $V = V_n$ considered as a commutative Lie group. Its Lie algebra is identified with the semidirect product of $\mathfrak{gl}(2, \mathbb{R})$ and V_n with a basis $\{X, Y, H, Z, E_0, \dots, E_n\}$. We fix a gradation of $\mathfrak{g}$, which will play an important role in the sequel:

$$\mathfrak{g}_1 = \langle Y \rangle; \quad \mathfrak{g}_0 = \langle H, Z \rangle; \quad \mathfrak{g}_{-1} = \langle X, E_n \rangle;$$
$$\mathfrak{g}_{-i} = \langle E_{n+1-i} \rangle \quad \text{for } 2 \leq i \leq n+1.$$

In the sequel by degree of elements of $\mathfrak{g}$ or any associated space we shall mean the degree corresponding to this gradation.

Let $\mathfrak{g}_-$ denote the negative part of $\mathfrak{g}$, which is a nilpotent subalgebra generated by $\mathfrak{g}_{-1}$. Let $\mathfrak{h}$ be the non-negative part of $\mathfrak{g}$ and H the corresponding subgroup in G. Then H can be identified with the subgroup of all lower-triangular matrices in $GL(2, \mathbb{R})$.

Our nearest goal is to construct a Cartan connection on $\mathcal{E}$, modelled by the homogeneous space G/H, that will be naturally associated with the equation (2). Such Cartan connection consists of a principal H-bundle $\pi : \mathcal{P} \to \mathcal{E}$ and the $\mathfrak{g}$-valued differential form ω on $\mathcal{P}$ such that

(1) $\omega(X^*) = X$ for all fundamental vector fields X^* on $\mathcal{P}$, $X \in \mathfrak{h}$;
(2) $R_h^*\omega = \operatorname{Ad} h^{-1}\omega$ for all $h \in H$;
(3) ω defines an absolute parallelism on $\mathcal{P}$.

Any $\mathfrak{g}$-valued form ω can be written as

$$\omega = \sum_{i=0}^{n} \omega^i E_i + \omega^x X + \omega^h H + \omega^z Z + \omega^y Y.$$

We say that a Cartan connection ω on a principal H-bundle $\pi : \mathcal{P} \to \mathcal{E}$ is *adapted to equation* (2), if for any local section s of π the set $\{s^*\omega^x, s^*\omega^0, \dots, s^*\omega^n\}$ is an adapted coframe on $\mathcal{E}$. It is easy to see that this definition does not depend on the choice of the local section s.

Denote by $C^k(\mathfrak{g}_-, \mathfrak{g})$ the space of all k-cochains on $\mathfrak{g}_-$ with values in $\mathfrak{g}$. Any Cartan connection ω modelled by the homogeneous space G/H determines the curvature tensor

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega]$$

on $\mathcal{P}$ and the curvature function $c : \mathcal{P} \to C^2(\mathfrak{g}_-, \mathfrak{g})$, where

$$c_p(u, v) = \Omega_p\big(\omega_p^{-1}(u), \omega_p^{-1}(v)\big)$$

for all $u, v \in \mathfrak{g}_-,\ p \in \mathcal{P}$.

This function satisfies the condition

$$c(ph) = h^{-1}.c(p) \quad \text{for all } h \in H,\ p \in \mathcal{P}, \tag{4}$$

where H acts on $C^2(\mathfrak{g}_-, \mathfrak{g})$ in the natural way.

Since $\mathfrak{g}_-$ and $\mathfrak{g}$ are graded, all spaces $C^k(\mathfrak{g}_-, \mathfrak{g})$ inherit the gradation

$$C^k(\mathfrak{g}_-, \mathfrak{g}) = \sum_r C^k_r(\mathfrak{g}_-, \mathfrak{g}),$$

where

$$C^k_r(\mathfrak{g}_-, \mathfrak{g}) = \big\{\alpha \in C^k(\mathfrak{g}_-, \mathfrak{g}) \big| \alpha(\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_k}) \subset \mathfrak{g}_{i_1+\dots+i_k+r}\big\}.$$

It is easy to show that the standard cochain differential

$$\partial : C^k(\mathfrak{g}_-, \mathfrak{g}) \to C^{k+1}(\mathfrak{g}_-, \mathfrak{g})$$

preserves this gradation. Using this gradation, we can decompose the curvature function c to the sum $c = \sum_r c_r$, where each c_r takes values in $C^2_r(\mathfrak{g}_-, \mathfrak{g})$.

There is another important decreasing filtration defined on the cochain complex $C^k(\mathfrak{g}_-, \mathfrak{g})$:

$$F^r C^k(\mathfrak{g}_-, \mathfrak{g}) = \Big\{\alpha \in C^k(\mathfrak{g}_-, \mathfrak{g}) \Big| \alpha(\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_k}) \subset \sum_{j \ge \min(i_1, \dots, i_k)} \mathfrak{g}_{j-k+r}\Big\}.$$

Let us note that

$$[\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}] \subset \mathfrak{g}_{i_1+i_2} \subset \sum_{j \ge \min(i_1, i_2)} \mathfrak{g}_{j-1}, \qquad \text{for all } i_1, i_2 \in \mathbb{Z}. \tag{5}$$

Indeed, this statement is trivial for $i_1 \ge -1$ or $i_2 \ge -1$, and $[\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}] = 0$ for $i_1, i_2 < -1$. Using (5) it is easy to show that the filtration $F^r C(\mathfrak{g}_-, \mathfrak{g})$ is preserved by the cochain differential. In the following we shall denote the intersection $F^r C^k(\mathfrak{g}_-, \mathfrak{g}) \cap C^k_s(\mathfrak{g}_-, \mathfrak{g})$ by $F^r C^k_s(\mathfrak{g}_-, \mathfrak{g})$.

Proposition 1. *For any adapted Cartan connection its curvature function satisfies the conditions*:

1. $c_r = 0$ *for all* $r \leq 0$;
2. $c \subset F^1 C^2(\mathfrak{g}_-, \mathfrak{g})$.

Proof. These conditions immediately follow from the formulas (3) for the adapted coframe. □

So, we see that the structure function of any adapted Cartan connection lies in the subspace

$$\mathcal{S} = \sum_{r>0} F^1 C^2_r(\mathfrak{g}_-, \mathfrak{g}). \tag{6}$$

2.3. Harmonic analysis on the symbol algebra

Generally speaking, there are many Cartan connections adapted to a given equation (2). The basic idea of choosing a unique one among them is to add linear conditions on structure function. Finding these conditions is not easy since they should guarantee the existence and uniqueness of the required Cartan connection and at the same time they must be invariant with respect to the action of H on $C^2(\mathfrak{g}_-, \mathfrak{g})$ because of property (4) of the curvature function c. Hopefully, as it was shown by T. Morimoto ([6, 7]), they can be derived from the "harmonic theory" on our Lie algebra $\mathfrak{g}$.

First, we fix a metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that vectors $E_0, \dots, E_n, X, H, Z, Y$ form an orthogonal basis and

$$\langle E_i, E_i \rangle = (n-i)!\,/i!, \quad 0 \leq i \leq n;$$
$$\langle X, X \rangle = \langle Y, Y \rangle = 1,$$
$$\langle H, H \rangle = \langle Z, Z \rangle = 2.$$

This metric $\langle \cdot, \cdot \rangle$ is chosen in such a way that

1. all spaces $\mathfrak{g}_i$ are mutually orthogonal;
2. $\langle S, T \rangle = \operatorname{tr} {}^tST$ for all $S, T \in \mathfrak{gl}(2, \mathbb{R})$;
3. $\langle Su, v \rangle = \langle u, {}^tSv \rangle$ for all $S \in \mathfrak{gl}(2, \mathbb{R})$, $u, v \in V$, so that the transposition with respect to this metric on V agrees with standard matrix transposition in $\mathfrak{gl}(2, \mathbb{R})$.

Then we extend this metric to the spaces $C^k(\mathfrak{g}_-, \mathfrak{g})$ in the standard way and denote by

$$\partial^* : C^{k+1}(\mathfrak{g}_-, \mathfrak{g}) \to C^k(\mathfrak{g}_-, \mathfrak{g})$$

the operator adjoint to the cochain differential ∂.

Finally, we add one more condition on our Cartan connection adapted to equation (2).

Proposition 2 ([4, 12]). *Among all Cartan connections adapted to equation* (2) *there exists a unique* (*up to isomorphism*) *Cartan connection whose structure function is coclosed, i.e.,* $\partial^* c = 0$.

In the sequel we call this Cartan connection *a canonical connection associated with equation* (2) and denote it by $\omega_{\mathcal{E}}$.

The canonical Cartan connection for third order ODE's was constructed H. Sato and A. Yoshikawa ([9]). Similar absolute parallelism structures were constructed by R. Bryant ([1]) and M. Fels ([5]) in case of 4 th order ODE. The same connection in case of linear differential equations was constructed by Se-ashi ([10]). Its structure function in this case is of particularly simple form. Namely, $c = c_3 + \cdots + c_{n+1}$, where c_i takes values in the one-dimensional subspace in $C^2(\mathfrak{g}_-, \mathfrak{g})$, generated by the mapping ϕ_i of degree i:

$$\phi_i(X, v) = Y^{i-1}v, \quad \phi_i(u, v) = 0, \qquad \text{for all } u, v \in V. \tag{7}$$

3. Fundamental invariants

Canonical Cartan connections give the direct and economic way of constructing invariants of ordinary differential equations. Let $\omega_{\mathcal{E}} : T\mathcal{P} \to \mathfrak{g}$ be a canonical Cartan connection associated with the equation $\mathcal{E}$. Since $\omega_{\mathcal{E}}$ defines an absolute parallelism on $\mathcal{P}$, we see that the algebra of its invariants is generated by coefficients of its structure function and their covariant derivatives. Using the special properties of the canonical Cartan connection and in particular its deep relation with harmonic analysis on the symbol algebra $\mathfrak{g}$, we may reduce the number of generators of the algebra of invariants of $\omega_{\mathcal{E}}$.

Proposition 3 ([4]). *The algebra of invariants of the canonical Cartan connection* $\omega_{\mathcal{E}}$ *is generated by the coefficients of the harmonic part of the structure function* c *of* $\omega_{\mathcal{E}}$ *and its covariant derivatives.*

In particular, the curvature function c *vanishes if and only if its harmonic part vanishes.*

So, in order to describe the number of generators in the algebra of invariants of the canonical Cartan connection, we need to compute the following space

$$\mathcal{F} = \{c \in \mathcal{S} | \partial c = \partial^* c = 0\}, \tag{8}$$

where $\mathcal{S}$ is given by (6).

Let $\mathfrak{a}$ be the image of $\mathfrak{gl}(2, \mathbb{R})$ in $\mathfrak{gl}(V)$ corresponding to the representation of $\mathfrak{gl}(2, \mathbb{R})$ in V. Let us recall the classical Spencer operator

$$S^k : \operatorname{Hom}\Big(\bigwedge^k V, \mathfrak{a}\Big) \to \operatorname{Hom}\Big(\bigwedge^{k+1} V, V\Big),$$

$$S^k(\phi)(v_1 \wedge v_2 \wedge \cdots \wedge v_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \phi(v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{k+1}) v_i.$$

Proposition 4.

1. *The space $\mathcal{F}$ can be decomposed into three subspaces $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$ where*

$$\mathcal{F}_1 = \Big\{\alpha \in \mathrm{Hom}(\mathbb{R}X \otimes V, V) \cap \mathcal{S} \Big| \big\langle i_X(\alpha), \mathfrak{a} + [X, \mathfrak{gl}(V)]\big\rangle = 0\Big\};$$

$$\mathcal{F}_2 = \Big\{\beta \in \mathrm{Hom}\Big(\bigwedge^2 V, V\Big) \cap \mathcal{S} \Big| X.\beta = 0 \text{ and } \langle \beta, \mathrm{im}\ S^1\rangle = 0\Big\};$$

$$\mathcal{F}_3 = \Big\{\gamma \in \mathrm{Hom}\Big(\bigwedge^2 V, \mathfrak{a}\Big) \subset \mathcal{S} \Big| S^2(\gamma) = 0 \text{ and } X.\gamma = 0\Big\}.$$

2. *For any $n \geq 3$ the subspace $\mathcal{F}_1$ is of dimension $n-1$ and is generated by elements ϕ_i of degree i, $i = 3, \dots, n+1$ given by equation* (7).

3. *The subspace $\mathcal{F}_2$ is trivial for $n = 3$, generated by one element of degree* 2 *for $n = 4$, by two elements of degrees* 2 *and* 3 *for $n = 5$, and by three elements of degrees* 2, 3, *and* 4 *for $n \geq 6$.*

4. *The subspace $\mathcal{F}_3$ is generated by two elements of degrees* 3 *and* 4 *for $n = 3$, by one element of degree* 6 *for $n = 4$, and is trivial for $n \geq 5$.*

Proof. 1. The spaces $C^k(\mathfrak{g}_-, \mathfrak{g})$ are decomposed into the sum of mutually orthogonal subspaces $\mathrm{Hom}(\mathbb{R}X \otimes \bigwedge^{k-1} V, V)$, $\mathrm{Hom}(\mathbb{R}X \otimes \bigwedge^{k-1} V, \mathfrak{a})$, $\mathrm{Hom}(\bigwedge^k V, V)$, and $\mathrm{Hom}(\bigwedge^k V, \mathfrak{a})$. It is easy to check that the cochain differential restricted to these subspaces has the form:

$$\partial\alpha = 0$$

for all $\alpha \in \mathrm{Hom}(\mathbb{R}X \otimes \bigwedge^{k-1} V, V)$;

$$\partial\alpha \in \mathrm{Hom}\Big(\mathbb{R}X \otimes \bigwedge^k V, V\Big) \quad \text{and} \quad i_X\partial\alpha = S^{k-1}(i_X\alpha)$$

for all $\alpha \in \mathrm{Hom}(\mathbb{R}X \otimes \bigwedge^{k-1} V, \mathfrak{a})$;

$$\partial\alpha \in \mathrm{Hom}\Big(\mathbb{R}X \otimes \bigwedge^k V, V\Big) \quad \text{and} \quad i_X\partial\alpha = X.\alpha$$

for all $\alpha \in \mathrm{Hom}(\bigwedge^k V, V)$;

$$\partial\alpha \in \mathrm{Hom}\Big(\mathbb{R}X \otimes \bigwedge^k V, \mathfrak{a}\Big) + \mathrm{Hom}\Big(\bigwedge^{k+1} V, V\Big)$$
$$\text{and } i_X\partial\alpha = X.\alpha,\ \partial\alpha|_{\bigwedge^{k+1} V} = S^k(\alpha)$$

for all $\alpha \in \mathrm{Hom}(\bigwedge^k V, \mathfrak{a})$.

Using these formulas, we easily get that

$$\mathcal{F}_1 = \mathcal{F} \cap \mathrm{Hom}(\mathbb{R}X \otimes V, V);$$

$$\mathcal{F}_2 = \mathcal{F} \cap \mathrm{Hom}\Big(\bigwedge^2 V, V\Big); \quad \mathcal{F}_3 = \mathcal{F} \cap \mathrm{Hom}\Big(\bigwedge^2 V, \mathfrak{a}\Big),$$

and $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$.

The proof of items 2–4 of the proposition is based on the explicit description of the $\mathfrak{gl}(2, \mathbb{R})$-module V_n and on elementary linear algebra. □

4. Parametric computation of invariants

Let us outline the computation of fundamental invariants corresponding to the space $\mathcal{F}$ described by Proposition 4 and give the proofs of Theorems 1 and 2.

Let $\omega_{\mathcal{E}} : \mathcal{P} \to \mathcal{E}$ be the canonical Cartan connection of equation (2). Choosing a (local) section $s : \mathcal{E} \to \mathcal{P}$ we may identify ω with the $\mathfrak{g}$-valued 1-form $s^*\omega$ on $T\mathcal{E}$. Any other section s' is uniquely determined by the function $g : \mathcal{E} \to H$ such that $s' = sg$. Then

$$(s')^*\omega = \text{Ad}\,(g^{-1})(s^*\omega) + g^{-1}dg. \tag{9}$$

We may write ω as

$$\omega = \sum_{i=0}^{n} \omega^i E_i + \omega^x X + \omega^y Y + \omega^z Z + \omega^h H.$$

where the forms $s^*\omega^x, s^*\omega^0, \dots, s^*\omega^n$ are given by (3). Using (9) it is easy to prove that there exists a unique section $s : \mathcal{E} \to \mathcal{P}$ such that the functions A and B are reduced to 1 and the function A^n_{n-1} to 0. These conditions uniquely determine the section s relative to the fixed coordinate system (x, y) on the plane. To simplify the notation, throughout this section we identify ω with $s^*\omega$ and $s^*\omega^\alpha$ with ω^α for all $\alpha = 0, \dots, n, x, y, z, h$.

Similar to (3) let us write the forms ω^α, $\alpha = y, z, h$, as

$$\omega^\alpha = A^\alpha_x \tau + \sum_{j=0} A^\alpha_j \theta^j.$$

Computing curvature coefficients of degree 1, which all vanish, we obtain

$$A^h_x = -\frac{1}{n+1} f_n, \qquad A^z_x = \frac{1}{n(n+1)} f_n,$$

$$A^i_{i-1} = \frac{i(n-i)}{n+1} f_n \qquad \text{for all } i = 1, \dots, n.$$

Let us find the expression of invariants corresponding to the subspace $\mathcal{F}_1$ from Proposition 4. Denote by (a^i_j) the matrix $1 + Y + Y^2 + \dots + Y^n = (1 - Y)^{-1}$. Then we have

$$\Omega^i = \sum_{j=0}^{i} a^i_j L_{i-j+1} \omega^x \wedge \omega^j \bmod \langle \omega^i \wedge \omega^j | 0 \le i, j \le n \rangle,$$

where $L_1 = L_2 = 0$ and L_i, $i = 3, \dots, n+1$, are some functions on $\mathcal{E}$ that represent the required invariants.

Expanding both parts we get

$$A^i_{j-1} - A^{i+1}_j + a^i_j L_{i-j+1} = \Pi^i_j, \quad 0 \le j < i \le n-1,$$
$$A^n_{j-1} + a^n_j L_{n-j+1} = -f_j + \Pi^n_j, \quad 0 \le j < n,$$

where

$$\Pi^i_j = -(A^i_j)_x - \big(nA^z_x + (n-2i)A^h_x\big)A^i_j + i(n+1-i)A^y_x A^{i-1}_j - \sum_{k=j+1}^{i-1} a^k_j L_{k-j} A^i_k.$$

In particular, taking into account that $L_2 = 0$, we find that

$$A^y_x = \frac{6}{n(n+1)(n+2)} f_{n-1} + \frac{n-1}{(n+1)(n+2)} f_{nx} - \frac{n-1}{(n+1)^2(n+2)} f_n^2,$$

and A^{i+1}_{i-1}, $i = 1, \dots, n-1$ are some linear combinations of terms f_{n-1}, f_{nx} and f_n^2.

Suppose for a fixed $2 \le k \le n$ we computed all invariants L_k and the coefficients A^i_j such that $i - j \ge k$. Then the formulas above can be considered as a set of linear equations on A^i_j with $i - j = k + 1$ and the invariant L_{k+1}. It is easy to see that this set of equations has a unique solution and enables to compute all invariants L_k recursively for $k = 3, \dots, n+1$.

Since the right part of these equations contains only polynomials of already known coefficients and invariants and expressions $(A^i_j)_x$ and f_i, we see that all invariants L_k are polynomials in $(d/dx)^i f_j$. In particular, these invariants are uniquely determined by their values on the set of linear differential equations.

Now we apply the result of Se-ashi ([10, 11]), who showed that the invariants L_i in case of linear differential equations coincide up to a constant with classical Wilczynski invariants. This proves Theorem 1.

Further parametric computation of curvature terms of degrees 2, 3 and 4 shows that the invariants corresponding to the subspace $\mathcal{F}_2$ have the form $I_2 = f_{nn}$ (for $n \ge 4$), $I_3 = f_{n,n-1} + (n(n-1)/(n+1)(n-2)) f_n f_{nn} + (n/(n-2)) f_{nnx}$ (for $n \ge 5$), and $I_4 = f_{n-1,n-1}$ modulo the differential ideal generated by I_2, I_3 and L_3 (for $n \ge 6$).

The invariants corresponding to the subspace $\mathcal{F}_3$ are

- for $n = 3$, $J_3 = f_{333}$ and $J_4 = 6 f_{233} + f_{33}^2$ modulo the differential ideal generated by J_3;
- for $n = 4$, $J_6 = 6 f_{234} - 4 f_{333} - 3 f_{34}^2$ modulo the differential ideal generated by I_2 and I_3.

This completes the proof of Theorem 2.

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