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On the local structure of A-jet spaces[1]

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Abstract. We analyze the local structure of A-jet spaces, where A is a Weil algebra; by the way, we introduce the bundles of A-jets of sections of a regular projection and describe their vertical tangent spaces.

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The spaces $J^k_m M$ of k-jets of m-dimensional submanifolds of a manifold M (also known as higher order Grassmann bundles) can be naturally thought of as certain spaces of ideals of $\mathcal{C}^\infty(M)$; this point of view was introduced by Muñoz, Muriel and Rodríguez, [7]; moreover, the spaces $J^k_m M$ are associated to Weil bundles induced by the Weil algebras $\mathbb{R}^k_n$ (see below). The same way, also a general concept of jet (or contact element) associated to any Weil algebra A can be defined; it was done, in [7], and also considered by Kolář, [3]; the smooth structure on that spaces was obtained, in [1], and a geometrical interpretation of A-jets was given by Kureš and Mikulski, [6]; on the other hand, in [2], a canonical immersion into Grassmann bundles is studied. Finally, for an extension to fibered manifolds of the concept of A-jet we refer to ([5]).

In this paper the local structure of A-jet spaces is analyzed, getting a few isomorphisms which extend well-known properties of the higher order Grassmann bundles $J^k_m M$. The main result, roughly speaking, locally reduces the structure of an arbitrary A-jet space to the case when the dimension of the manifold M equals the width of A (see below). As a previous step, we introduce the notion of A-jets of local sections which play an analogous role to that of ordinary jets of sections and we also describe their vertical tangent spaces in terms of derivations of $\mathcal{C}^\infty(M)$, following the ideas in ([7]).

[1] This paper is in final form and no version of it will be submitted for publication elsewhere. Partially founded by Junta de Castilla y León under contract SA30/00B.

1. Weil algebras and Weil bundles

In this section we will recall the main notions and results on Weil bundles, which will be useful later on.

Definition 1.1. An $\mathbb{R}$-algebra A is called a *Weil algebra* if it is finite-dimensional, local and rational. Let us denote by $\mathfrak{m}_A$ the maximal ideal of A. The integer k such that $\mathfrak{m}_A^{k+1} = 0$, $\mathfrak{m}_A^k \neq 0$, will be called the *order* of A and denoted by $o(A)$. The dimension of $\mathfrak{m}_A/\mathfrak{m}_A^2$ will be called the *width* of A and denoted by $w(A)$.

The main examples of Weil algebras are the rings of truncated polynomials: for given integers n and k, we define $\mathbb{R}_n^k := \mathbb{R}[\epsilon_1, \ldots, \epsilon_n]/\mathfrak{m}^{k+1}$ where the ϵ's are undetermined variables and $\mathfrak{m}$ is the maximal ideal that they generate; then $\mathbb{R}_n^k$ is a Weil algebra with $o(\mathbb{R}_n^k) = k$ and $w(\mathbb{R}_n^k) = n$ (when $k = n = 1$ we obtain the ring of dual numbers $\mathbb{R}_1^1$).

On the other hand, if $\mathfrak{m}_p$ denotes the maximal ideal associated to a point p in a manifold M, the quotients $\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1}$ are also Weil algebras isomorphic to $\mathbb{R}_n^k$ where n is the dimension of M (an isomorphism is induced by taking local coordinates).

There is a number of equivalent definitions of Weil algebra (see [4]); in fact, any Weil algebra A can be always obtained as a quotient of a suitable $\mathbb{R}_n^k$ (necessarily $n \geq w(A)$ and $k \geq o(A)$) and conversely, such quotients are Weil algebras.

For each integer $j \leq o(A)$, it can be defined a new algebra $A_j \overset{\text{def.}}{=} A/\mathfrak{m}_A^{j+1}$ which will be called the j-th *underlying algebra* of A (see [3], where this notion was considered). We have $A_{o(A)} = A$, $A_1 \simeq \mathbb{R}^1_{w(A)}$ and $A_0 = \mathbb{R}$. On the other hand each $\mathbb{R}$-algebra morphism $\psi : A \longrightarrow B$ between Weil algebras naturally induces morphisms $\psi_j : A_j \longrightarrow B_j$.

Later on we will use the following result:

Lemma 1.2. *Let A, B be Weil algebras and $\psi : A \longrightarrow B$ an $\mathbb{R}$-algebra morphism. The following assertions are equivalent,*
(i) *$\psi : A \longrightarrow B$ is epimorphic.*
(ii) *$\psi_1 : A_1 \longrightarrow B_1$ is epimorphic.*
(iii) *$\psi_1 : \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is epimorphic.*

Proof. (i)$\Longleftrightarrow$(ii) The surjectivity of ψ_1 follows trivially from that of ψ. Conversely, let us suppose that ψ_1 is an epimorphism and let $\{b_1, \ldots, b_r\}$ be a set of linearly independent generators of $\mathfrak{m}_B \bmod \mathfrak{m}_{\mathrm{B}}^2$. We are looking for preimages of b_i, $i = 1, \ldots, r$ by ψ. From the surjectivity of ψ_1 follows the existence of elements $a_i \in A$ such that $b_i - \psi(a_i) \in \mathfrak{m}_B^2$; so if $k = o(B)$ we have $b_i - \psi(a_i) \in \mathfrak{m}_B^2$, $b_i b_j - \psi(a_i a_j) \in \mathfrak{m}_B^3, \ldots, b_{i_1} b_{i_2} \cdots b_{i_k} - \psi(a_{i_1} a_{i_2} \cdots a_{i_k}) = 0$. By substitution from the bottom to the top, we get finally polynomials $P_i = P_i(a_1, \ldots, a_r) \in A$ such that $b_i = \psi(P_i)$.

(ii)$\Longleftrightarrow$(iii) It is obvious. □

It is well known that a manifold M can be recovered as the set of its $\mathbb{R}$-points, which are the morphisms $\mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$; also the tangent bundle TM is obtained by taking the 'points' with values in the dual numbers, $\mathcal{C}^\infty(M) \longrightarrow \mathbb{R}^1_1$. In general we can consider 'points' of M taking values in an algebra A. This concept comes back to Weil, [10], who called them 'points A-proches' of M.

Definition 1.3. Let M be a manifold and A a Weil algebra. An $\mathbb{R}$-algebra morphism

$$p^A : \mathcal{C}^\infty(M) \longrightarrow A$$

is called an *A-point* (or *A-velocity*) of M. The set of A-points of M will be called the *Weil bundle of A-points* of M and denoted by M^A.

To simplify notation, when $A = \mathbb{R}^k_n$ we will write M^k_n instead of $M^{\mathbb{R}^k_n}$. This way, $M^0_n = M$ (for any n) and $M^1_1 = TM$.

Definition 1.4. We will say that an A-point p^A is *regular* if it is surjective; the set of regular A-points of M will be denoted by $\check{M}^A$.

Remark. Let ϕ be an immersion of a neighborhood of $0 \in \mathbb{R}^n$ into a manifold M; the composition

$$\mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{m}^{k+1}_{\phi(0)} \xrightarrow{\phi^*} \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{m}^{k+1}_0 = \mathbb{R}^k_n$$

is a regular $\mathbb{R}^k_n$-point. This way, the concept of infinitesimal parametrized submanifold (at the order k) is embedded into that of regular A-point.

If we compose an A-point p^A with the canonical projection $A \longrightarrow A/\mathfrak{m}_A = \mathbb{R}$ we obtain an $\mathbb{R}$-point of M, that is, an ordinary point $p \in M$; this defines a projection $M^A \longrightarrow M$. If $k = o(A)$, then p^A vanishes on $\mathfrak{m}^{k+1}_p$ and so we get a morphism between Weil algebras, $\mathcal{C}^\infty(M)/\mathfrak{m}^{k+1}_p \longrightarrow A$ which we will also denote by p^A because it determines p^A.

Each function $f \in \mathcal{C}^\infty(M)$ defines a map $f^A : M^A \longrightarrow A$, by the rule

$$f^A(p^A) \stackrel{\text{def}}{=} p^A(f).$$

Theorem 1.5. *There exists a differentiable structure on M^A determined by the condition that the maps f^A are smooth; moreover, $\check{M}^A$ is an open set of M^A. Furthermore, $M^A \longrightarrow M$ is a fiber bundle with typical fiber* $\mathrm{Hom}(\mathbb{R}^k_n, A)$, *where $n = \dim M$ and $k = o(A)$.*

Proof. For a proof, see [4, 7]. □

Definition 1.6. Let $\psi : A \longrightarrow B$ be a morphism of Weil algebras. For each manifold M we have an induced smooth map of fiber bundles

$$\psi_M : M^A \longrightarrow M^B$$

defined by the rule:

$$\psi_M(p^A) \stackrel{\text{def}}{=} \psi \circ p^A, \qquad p^A \in M^A.$$

Definition 1.7. Let $\phi : M \longrightarrow N$ be an smooth map between the manifolds M and N. For each Weil algebra A we have an induced smooth map

$$\phi^A : M^A \longrightarrow N^A$$

defined by the rule: $\phi^A(p^A) \overset{\text{def}}{=} p^A \circ \phi^*$, $p^A \in M^A$, where ϕ^* stands for the map induced between the rings of functions of M and N.

Theorem 1.8. *There is a natural identification*

$$T_{p^A} M^A \simeq \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), A),$$

where each $X \in T_{p^A} M^A$ is related to the derivation $X' \in Der_{\mathbb{R}}(\mathcal{C}^\infty(M), A)$ determined by $X'(f) = X(f^A) \in A$, $f \in \mathcal{C}^\infty(M)$, X derives component-wise the vector-valued function f^A and A is considered as a $\mathcal{C}^\infty(M)$-module via p^A, see [7].

2. A-jet manifolds

In this section we will recall the main results in ([1]); their proofs will be just sketched.

The concept of infinitesimal parametrized submanifold (of order k) is recovered in the notion of regular $\mathbb{R}^k_n$-point p^k_n. In order to eliminate the 'parametrization' we have to consider as equivalent both p^k_n and its composition with any automorphism of $\mathbb{R}^k_n$; alternatively, we can take the kernel of p^k_n.

Definition 2.1. Given a manifold M, the kernel of a regular A-point p^A will be called the *jet* of p^A and we will denote it by $\mathfrak{p}^A = \ker(p^A)$. The set of jets of regular A-points will be called the *space of A-jets* of M and denoted by $J^A M$; thus, we have a surjective map $\ker : \check{M}^A \longrightarrow J^A M$ which associates to each A-point its kernel.

By the definition, for each $\mathfrak{p}^A \in J^A M$ we have $\mathcal{C}^\infty(M)/\mathfrak{p}^A \simeq A$.

The group $\mathrm{Aut}(A)$ acts on $\check{M}^A$ by composition: given $p^A \in \check{M}^A$ and $g \in \mathrm{Aut}(A)$ we define $g \cdot p^A \overset{\text{def.}}{=} g_M(p^A) = g \circ p^A$ (Definition 1.6). There is an obvious equivalence between the set of orbits of this action and $J^A M$.

From now on in this section, we will fix an n-dimensional manifold M, a Weil algebra A with $o(A) = k$, $w(A) = m$ and an epimorphism $\alpha : \mathbb{R}^k_n \longrightarrow A$. So, we have a map of fiber bundles $\alpha_M : \check{M}^k_n \longrightarrow \check{M}^A$, where $\alpha_M(p^k_n) = \alpha \circ p^k_n$ (see Definition 1.6).

Each coordinate open set $U \subset M$ trivializes simultaneously both the bundles $\check{M}^k_n$ and $\check{M}^A$: let $\{y_1, \dots, y_n\}$ be a system of coordinates on U; for each point $p \in U$ we have the following isomorphism,

$$\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} \overset{s_U(p)}{\simeq} \mathbb{R}^k_n, \qquad y_i \mapsto y_i(p) + \epsilon_i$$

and then, if we denote the group $\mathrm{Aut}(\mathbb{R}^k_n)$ by G, we get the trivialization of $\check{M}^k_n$ as

$$\check{U}^k_n \longrightarrow U \times G, \quad p^k_n \mapsto (p, g) \text{ with } g \circ s_U(p) = p^k_n$$

and the trivialization of $\check{M}^A$ as

$$\check{U}^A \longrightarrow U \times G/G_\alpha, \quad p^A \mapsto (p, [g]_{G_\alpha}) \text{ with } \alpha \circ g \circ s_U(p) = p^A$$

where G_α is the closed subgroup of G comprised by the elements leaving α invariant and $[g]_{G_\alpha}$ is the class of $g \bmod G_\alpha$.

In terms of the above trivializations, the map α_M is just the factor map by G_α. Now, the projection $\ker : \check{M}^A \longrightarrow J^A M$ gives bijective maps

$$\ker(\check{U}^A) \xrightarrow{\Phi} U \times G/G_{\ker(\alpha)}$$

where $G_{\ker(\alpha)}$ is the closed subgroup of G comprised by the elements leaving $\ker(\alpha)$ invariant. The bijections Φ can be used to define on $J^A M$ a smooth structure with which the map ker becomes the factor map under $\mathrm{Aut}(A)$. Observe that $G_{\ker(\alpha)}/G_\alpha \simeq \mathrm{Aut}(A)$; so,

Theorem 2.2. *On $J^A M$ there exists a smooth structure such that* $\ker : \check{M}^A \longrightarrow J^A M$ *is a principal fiber bundle with group* $\mathrm{Aut}(A)$.

Remark. When $A = \mathbb{R}^k_m$, we obtain the well-known spaces of k-jets of m-dimensional submanifolds of M, $J^k_m(M)$ (also called higher order Grassmann bundles, see [7, 9]).

The tangent space of $\check{M}^A$ at p^A projects onto that of $J^A M$ at $\mathfrak{p}^A$; therefore, $T_{\mathfrak{p}^A} J^A M$ is a quotient space of $T_{p^A} M^A \simeq \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), A)$ (see Theorem 1.8). By definition, the vector functions f^A, $f \in \mathfrak{p}^A$, vanish on the fiber $(\ker)^{-1}(\mathfrak{p}^A)$; it follows that the vertical tangent space $T^v_{p^A} M^A \subset \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), A)$ kills $\mathfrak{p}^A$, thus that space can be identified with a subset of $\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M)/\mathfrak{p}^A, A)$; but the Lie algebra of $\mathrm{Aut}(A)$ is $\mathrm{Der}_{\mathbb{R}}(A, A)$ and this clearly forces

$$T^v_{p^A} M^A \simeq \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M)/\mathfrak{p}^A, A).$$

Finally, by using p^A we have $\mathcal{C}^\infty(M)/\mathfrak{p}^A \simeq A$ and then,

Theorem 2.3. *For each $\mathfrak{p}^A \in J^A M$, the space $T_{\mathfrak{p}^A} J^A M$ is isomorphic to*

$$\mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\mathfrak{p}^A\big)/\,\mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(M)/\mathfrak{p}^A, \mathcal{C}^\infty(M)/\mathfrak{p}^A\big).$$

On the other hand, if $\mathfrak{p}^A \in J^A M$ projects onto $p \in M$, then $\mathfrak{m}^{k+1}_p \subseteq \mathfrak{p}^A \subseteq \mathfrak{m}_p$. This way, $\mathfrak{p}^A$ is identified with a d-plane of $\mathfrak{m}_p/\mathfrak{m}^{k+1}_p$ (the fiber of the k-th tangent bundle $T^{*,k}M$ at $p \in M$) with $d = \dim(\ker(\alpha))$.

Theorem 2.4. *The space of A-jets, $J^A M$, is a submanifold of the grassmannian* $\mathrm{Gr}(d, T^{*,k}M)$. *Moreover, this inclusion is a morphism of fiber bundles on M. For an analysis of this immersion, see* [2].

3. Induced maps on A-jet spaces

Each smooth map $\phi : M \longrightarrow N$ induces a new map between the corresponding A-Weil bundles, $\phi^A : M^A \longrightarrow N^A$ which sends $p^A \in M^A$ to $\phi^A(p^A) \overset{\text{def.}}{=} p^A \circ \phi^*$ (see Definition 1.7). However, the condition of regularity of an A-point is not, in general, preserved, that is, $\phi^A(\check{M}^A) \not\subseteq \check{N}^A$. This is why we give the following

Definition 3.1. Let $\phi : M \longrightarrow N$ be a differentiable map; an A-point $p^A \in M^A$ will be called *ϕ-regular* if $\phi^A(p^A) = p^A \circ \phi^* \in \check{N}^A$. The set of ϕ-regular A-points of M^A will be denoted by $\check{M}^A_\phi$.

It follows from Lemma 1.2 that a p^A is ϕ-regular if and only if the composition

$$T^*_{\phi(p)}N = \mathfrak{m}_{\phi(p)}/\mathfrak{m}_{\phi(p)^2} \xrightarrow{\phi^*} T^*_pM = \mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{(p^A)_1} \mathfrak{m}_A/\mathfrak{m}_A^2$$

is an epimorphism, where $(p^A)_1$ denotes the map induced by p^A; in particular,

Proposition 3.2. *The set of ϕ-regular A-points, $\check{M}^A_\phi$, is an open subset of M^A (eventually the empty set).*

Definition 3.3. The set of jets of ϕ-regular A-points will be denoted by $J^A_\phi M$.

The condition of regularity is preserved by the action of the group $\mathrm{Aut}(A)$; furthermore, $J^A_\phi M$ is an open subset of $J^A M$ and so, $\check{M}^A_\phi \xrightarrow{\ker} J^A_\phi M$ is a principal fiber bundle.

Proposition 3.4. *The map $\phi : M \longrightarrow N$ induces maps*

$$\check{M}^A_\phi \xrightarrow{\phi^A} \check{N}^A, \quad p^A \mapsto \phi^A(p^A) \overset{\text{def.}}{=} p^A \circ \phi^*$$
$$J^A_\phi M \xrightarrow{j^A\phi} J^A N, \quad \mathfrak{p}^A \mapsto j^A\phi(\mathfrak{p}^A) \overset{\text{def.}}{=} \ker(p^A \circ \phi^*) = (\phi^*)^{-1}\mathfrak{p}^A$$

which make commutative the following diagram:

$$\begin{array}{ccc} \check{M}^A_\phi & \xrightarrow{\phi^A} & N^A \\ {\scriptstyle \ker}\downarrow & & \downarrow{\scriptstyle \ker} \\ J^A_\phi M & \xrightarrow{j^A\phi} & J^A N. \end{array}$$

Examples. 1) If $\gamma : M \longrightarrow N$ is an immersion, then $\gamma^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M)$ is surjective on germs; so $\check{M}^A_\gamma = \check{M}^A$ and $J^A_\gamma M = J^A M$. In particular, J^A defines a functor in the category of differentiable manifolds with immersions (see [3]).

2) If $\pi : M \longrightarrow X$ is a fiber bundle and s is a section of π, then we have induced maps π^A, s^A, $j^A\pi$ and $j^A s$ such that $\pi^A \circ s^A = \mathrm{id}_{\check{X}^A}$ and $j^A\pi \circ j^A s = \mathrm{id}_{J^A X}$.

4. A-jets of local sections

Following with the second example in the above section, let us consider a fiber bundle $\pi : M \longrightarrow X$ and a Weil algebra A.

Definition 4.1. When $\dim X = w(A)$, the manifold

$$J^A(M/X) \stackrel{\text{def.}}{=} J^A_\pi M$$

will be called the *space of A-jets of local sections* of π.

The name assigned to the manifold $J^A(M/X)$ is due to the next property.

Proposition 4.2. *Given an A-jet* $\mathfrak{p}^A \in J^A(M/X)$ *which is projected onto* $\mathfrak{q}^A \in J^A X$*, there exists a local section* σ *of* π *such that* $\mathfrak{p}^A = (j^A\sigma)\mathfrak{q}^A$.

Proof. Let $m = \dim X = w(A)$, $n = \dim M \geq m$ and consider the diagram

$$\begin{array}{ccc} \mathbb{R}^k_n & \xrightarrow{\lambda} & \mathbb{R}^k_m \\ & {\scriptstyle\alpha'}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & A & \end{array}$$

where α is a surjective morphism, $\alpha' \stackrel{\text{def.}}{=} \alpha \circ \lambda$ and λ is the projection defined as $\lambda(\epsilon_i) = \epsilon_i$ if $i \leq m$ and $\lambda(\epsilon_i) = 0$, elsewhere.

Given $\mathfrak{p}^A \in J^A M$ which projects onto $\mathfrak{q}^A \in J^A X$ and $p \in M$, let us consider local charts $\{y_1, \dots, y_n\}$ and $\{x_1, \dots, x_m\}$ in neighborhoods of $p \in M$ and $x = \pi(p) \in X$ respectively, such that $\pi^* x_i = y_i$.

The above chosen system of local coordinates gives the isomorphisms

$$\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} \stackrel{s_M}{\simeq} \mathbb{R}^k_n, \qquad y_i \mapsto \epsilon_i,\ i = 1, \dots, n.$$

$$\mathcal{C}^\infty(X)/\mathfrak{m}_x^{k+1} \stackrel{s_X}{\simeq} \mathbb{R}^k_m, \qquad x_j \mapsto \epsilon_j,\ j = 1, \dots, m.$$

The relation between s_M and s_X is given by the commutativity of the following square,

$$\begin{array}{ccc} \mathbb{R}^k_m & \xrightarrow{\mu} & \mathbb{R}^k_n \\ {\scriptstyle s_X}\downarrow\wr & & \wr\downarrow{\scriptstyle s_M} \\ \mathcal{C}^\infty(X)/\mathfrak{m}_x^{k+1} & \xrightarrow[\pi^*]{} & \mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} \end{array}$$

where μ is the injection defined by $\mu(\epsilon_j) = \epsilon_j$, $j = 1, \dots, m$.

Now, if $\mathfrak{p}^A = \ker(p^A)$ for a suitable $p^A \in M^A$ there exists an automorphism $g \in \mathrm{Aut}(\mathbb{R}^k_n)$ making the diagram

$$\begin{array}{ccc} \mathbb{R}^k_n & \xrightarrow[\sim]{g} & \mathbb{R}^k_n \\ {\scriptstyle s_M}\downarrow \wr & & \downarrow {\scriptstyle \alpha'} \\ \mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} & \xrightarrow[p^A]{} & A \end{array}$$

commutative.

This way, the regularity condition on p^A is equivalent to the composition

$$\mathbb{R}^k_m \xrightarrow{\mu} \mathbb{R}^k_n \xrightarrow{g} \mathbb{R}^k_n \xrightarrow{\lambda} \mathbb{R}^k_m$$

to be an epimorphism and, therefore, an automorphism h of $\mathbb{R}^k_m$.

Let us take a section σ such that σ^* corresponds to $h^{-1} \circ \lambda \circ g$ via the isomorphisms s_M and s_X; that is $s_X \circ \sigma^* = (h^{-1} \circ \lambda \circ g) \circ s_M$.

If we define q^A to be the composition

$$\mathcal{C}^\infty(X)/\mathfrak{m}_x^{k+1} \overset{s_X}{\simeq} \mathbb{R}^k_m \xrightarrow{h} \mathbb{R}^k_m \xrightarrow{\alpha} A,$$

it can be easily checked that $\sigma^A(q^A) = p^A$.

So finally we have $\mathfrak{p}^A = \ker(\sigma^A(q^A)) = (j^A\sigma)\mathfrak{q}^A$. $\square$

Remark. When $A = \mathbb{R}^k_m$ we recover the well-known spaces of k-jets of local sections $J^k(M/X) = J^A(M/X)$ (see [7,8]). In this case, $J^AX = J^k_mX$ is identified with X because to each point $x \in X$ corresponds a unique jet, the power $\mathfrak{m}_x^{k+1}$; this is not true for a general Weil algebra A, [7].

5. The local structure of J^AM

Let $\mathfrak{p}^A = \ker(p^A) \in J^AM$, $p^A \in \check{M}^A$, which projects onto $p \in M$. The regularity of p^A means that

$$T_p^*M = \mathfrak{m}_p/\mathfrak{m}_p^{k+1} \xrightarrow{(p^A)_1} \mathfrak{m}_A/\mathfrak{m}_A^2,$$

is surjective (see Lemma 1.2). One deduces the existence of an m-dimensional subspace E of T_p^*M such that the restriction $(p^A)_1$ to E is an isomorphism; it follows that we can choose a manifold V and an m-dimensional submanifold $X \subset M$ such that, locally, $M = X \times V$ and $\pi^*T_x^*X = E$, where π is the projection of M onto the factor X. Therefore, the composition

$$\mathcal{C}^\infty(X) \xrightarrow{\pi^*} \mathcal{C}^\infty(M) \xrightarrow{p^A} A$$

is a regular A-point of X; that is, p^A is π-regular. The same property holds in a neighborhood of p^A; so,

Lemma 5.1. *Locally, J^AM takes the form $J^A(M/X)$ for a suitable projection $\pi : M \longrightarrow X$, with* $\dim X = w(A)$.

By this reason, when studying the local structure of $J^A M$ we can equivalently study the spaces of jets of sections $J^A(M/X)$; moreover, we can restrict ourselves to the case of a trivial projection $\pi : M = X \times V \longrightarrow X$.

Lemma 5.2. *Let $M = X \times V$ and $p = (x, v) \in M$. Then,*

(i) *Each A-point $p^A \in M^A$ projecting onto $p \in M$ defines by restriction to $\mathcal{C}^\infty(X)$ and to $\mathcal{C}^\infty(V)$, respectively, a couple $(q^A, r^A) \in X^A \times V^A$, where q^A projects onto $x \in X$ and r^A onto $v \in V$; conversely, each couple (q^A, r^A) as above defines a unique A-point in M^A, which will be denoted by $\zeta(q^A, r^A)$.*

(ii) *Given an A-jet $\mathfrak{p}^A \in J^A M$ onto p and an A-point $q^A \in X^A$, with $(j^A\pi)\mathfrak{p}^A = \mathfrak{q}^A = \ker(q^A)$, there exists a unique $p^A \in M^A$ such that $\ker(p^A) = \mathfrak{p}^A$ and $\pi^A(p^A) = q^A$.*

Proof. (i) It follows from the well-known isomorphism $X^A \times V^A \overset{\zeta}{\simeq} (X \times V)^A$, see [4, 7, 10]. The point $\zeta(q^A, r^A)$ is obtained in the following way: let us choose local charts $\{x_1, \dots, x_m\}$ and $\{v_1, \dots, v_h\}$ on X and V respectively and such that $x_j(x) = v_i(v) = 0$; so, $\{x_1, \dots, x_m, v_1, \dots, v_h\}$ is a local chart on M centered on p; then, $\zeta(q^A, r^A)$ is determined by the conditions

$$\zeta(q^A, r^A)(x_j) = q^A(x_j), \quad j = 1, \dots, m,$$
$$\zeta(q^A, r^A)(v_i) = r^A(v_i), \quad i = 1, \dots, h.$$

(ii) Let $\mathfrak{p}^A$, q^A and $\mathfrak{q}^A$ be as in the claim; the required A-point p^A is obtained by taking the composition

$$\mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{p}^A \overset{(\pi^*)^{-1}}{\simeq} \mathcal{C}^\infty(X)/\mathfrak{q}^A \xrightarrow{q^A} A;$$

Conversely, given p^A with $p^A|_{\mathcal{C}^\infty(X)} = q^A$ and $\ker(p^A) = \mathfrak{p}^A$ the previous composition coincides necessarily with p^A. □

Theorem 5.3. *Each section δ of $\check{X}^A \longrightarrow J^A X$ defines an isomorphism*

$$J^A(M/X) \simeq J^A X \times V^A.$$

Proof. Retaining the notation used in the above lemmas, we define a map Ψ by

$$J^A X \times V^A \xrightarrow{\Psi} J^A(M/X), \quad (\mathfrak{q}^A, r^A) \overset{\Psi}{\longmapsto} \ker\Big(\zeta\big(\delta(\mathfrak{q}^A), r^A\big)\Big).$$

From Lemma 5.2 it follows that Ψ is bijective; the smoothness of Ψ follows from that of δ, ζ and ker. □

Finally, from Lemma 5.1 and the above theorem we arrive to

Theorem 5.4. *The following isomorphism locally holds*

$$J^A M \simeq J^A X \times V^A,$$

where X and V are suitable manifolds with $\dim X = w(A)$ *and* $\dim V = \dim M - \dim X$.

Remark. When $A = \mathbb{R}^k_m$ we recover the known local isomorphism $J^k_m M \simeq X \times V^A$, because, in this case, $J^A X = X$ (see [7]).

6. Vertical tangent structures on $J^A(M/X)$

Let us consider the projection $J^A(M/X) \xrightarrow{j^A\pi} J^A X$ and let us fix $\mathfrak{p}^A$ with $(j^A\pi)\mathfrak{p}^A = \mathfrak{q}^A$. Thus, we have the usual exact sequence of tangent spaces

$$0 \longrightarrow T^v_{\mathfrak{p}^A} J^A(M/X) \longrightarrow T_{\mathfrak{p}^A} J^A(M/X) \xrightarrow{(j^A\pi)_*} T_{\mathfrak{q}^A} J^A(X) \longrightarrow 0.$$

Moreover, $T_{\mathfrak{p}^A} J^A(M/X) = T_{\mathfrak{p}^A} J^A(M)$ because $J^A(M/X)$ is an open set of $J^A(M)$ (Proposition 3.2) and we know (Theorem 2.3) the identification

$$T_{\mathfrak{p}^A} J^A M \simeq \mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\mathfrak{p}^A\big) / \mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(M)/\mathfrak{p}^A, \mathcal{C}^\infty(M)/\mathfrak{p}^A\big),$$

and the isomorphism $\mathcal{C}^\infty(X)/\mathfrak{q}^A \overset{\pi^*}{\simeq} \mathcal{C}^\infty(M)/\mathfrak{p}^A$ (see Definition 2.1); so,

$$T_{\mathfrak{p}^A} J^A(M/X) \simeq \mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(X)/\mathfrak{q}^A\big) / \mathrm{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(X)/\mathfrak{q}^A, \mathcal{C}^\infty(X)/\mathfrak{q}^A\big),$$

where $\mathcal{C}^\infty(X)/\mathfrak{q}^A$ is considered as an $\mathcal{C}^\infty(M)$-module by means of π^*.

On the other hand, the set $\mathrm{Der}_{\mathcal{C}^\infty(X)}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(X)/\mathfrak{q}^A)$, (derivations vanishing on $\mathcal{C}^\infty(X) \subset \mathcal{C}^\infty(M)$), intersects $\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(X)/\mathfrak{q}^A, \mathcal{C}^\infty(X)/\mathfrak{q}^A)$ just in 0 and then we have an inclusion

$$\mathrm{Der}_{\mathcal{C}^\infty(X)}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(X)/\mathfrak{q}^A\big) \subset T_{\mathfrak{p}^A} J^A(M/X).$$

It is obvious that this subset is comprised by vertical derivations in $T^v_{\mathfrak{p}^A} J^A(M/X)$.

Theorem 6.1. *With the same notation, we have a natural identification*

$$T^v_{\mathfrak{p}^A} J^A(M/X) \simeq \mathrm{Der}_{\mathcal{C}^\infty(X)}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(X)/\mathfrak{q}^A\big).$$

Proof. It is sufficient to consider the above mentioned inclusion and computing dimensions. First, by taking into account the local structure of $J^A(M/X)$ (Theorem 5.3), we see that $\dim T^v_{\mathfrak{p}^A} J^A(M/X) = \dim V^A$; second, we have

$$\dim \mathrm{Der}_{\mathcal{C}^\infty(X)}\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(X)/\mathfrak{q}^A\big) = \dim \mathrm{Der}_{\mathcal{C}^\infty(X)}\big(\mathcal{C}^\infty(M), A\big),$$

because $\mathcal{C}^\infty(X)/\mathfrak{q}^A \simeq A$, but the last one equals $\dim \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(V), A) = \dim V^A$ (see Theorem 1.8). Therefore, both dimensions coincide. □

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