Differential Geometry and Its Applications
Proc. Conf., Opava (Czech Republic), August 27–31, 2001
Silesian University, Opava, 2001, 43–48

On holomorphically projective mappings onto almost Hermitian spaces[1]

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Abstract. In this paper we consider holomorphically projective mappings from equiaffine spaces onto almost Hermitian spaces. We found the equations of these mappings in the form of a system of linear Cauchy equations. These results generalize the results obtained for holomorphically projective mappings of Kählerian spaces by J. Mikeš and analogous results about K-spaces and H-spaces obtained by I.N. Kurbatova.

We continue the investigation of F-planar mappings onto Hermitian and Riemannian spaces.

Keywords. Holomorphically projective mapping, equiaffine spaces, Riemannian spaces, affine-connected spaces, Hermitian spaces

MS classification. 53B20, 53B30, 53B35

1. Introduction

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers are devoted to geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings (see [1]–[3], [5]–[18]).

On the other hand, one line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces.

In this paper, we present some new results obtained for holomorphically projective mappings from equiaffine spaces A_n onto almost Hermitian spaces $\bar{H}_n$, which is not Kählerian.

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.
Supported by grant No. 201/02/0616 of The Grant Agency of Czech Republic.

By a *Hermitian space* H_n we mean a (pseudo-) Riemannian manifold together with an affinor structure F_i^h satisfying the conditions

$$(1) \qquad F_\alpha^h F_i^\alpha = -\delta_i^h, \quad g_{\alpha(i} F_{j)}^\alpha = 0$$

hold. Here g_{ij} is the metric tensor on H_n, δ_i^h is the Kronecker symbol and (i, j) denotes a symmetrization without normalization.

A natural classification contains 16 types of Hermitian spaces has been done by A. Gray and L.M. Hervella, see [4]. *Kählerian spaces* K_n are special cases of Hermitian spaces, which have a covariantly constant structure F_i^h.

In many papers holomorphically projective mappings and transformations of Hermitian spaces $H_n \to \bar{H}_n$ are studied (for example see [2, 5, 6, 9, 12, 15, 18]). These are special cases of F_1-planar mappings. In [7, 9], F_1-planar mappings from the space A_n with affine connection onto a Riemannian space $\bar{V}_n$ are defined and studied. These are characterized with respect to a common coordinate system x by the following equations

$$(2) \qquad \begin{aligned} &\text{a)} \quad \bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \delta_{(i}^h \psi_{j)} + F_{(i}^h \varphi_{j)}, \\ &\text{b)} \quad \bar{g}_{\alpha(i} F_{j)}^\alpha = 0, \end{aligned}$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the objects of affine connection on A_n and $\bar{V}_n$, respectively, $\bar{g}_{ij}$ is the metric tensor of $\bar{V}_n$, $\psi_i(x)$, $\varphi_i(x)$ are covectors, and

$$F_i^h(x) \; (\text{Rank} \, \|F_i^h - \rho\delta_i^h\| > 1)$$

is the affinor structure on A_n and $\bar{V}_n$. Equations (2) are equivalent to the equations

$$(3) \qquad \begin{aligned} &\text{a)} \quad \bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_{(i} \bar{g}_{j)k} + \varphi_{(i} \bar{F}_{j)k}, \\ &\text{b)} \quad \bar{g}_{\alpha(i} F_{j)}^\alpha = 0, \end{aligned}$$

where

$$\bar{F}_{ij} \stackrel{\text{def}}{=} \bar{g}_{i\alpha} F_j^\alpha.$$

Here and in what follows, comma denotes covariant derivative on A_n.

In [7], it is proved that a general solution of the system (3) for a given space A_n and a given structure F_i^h depends on finitely many parameters.

An F_1-planar mapping is called F_2-*planar* if the covector ψ_i is a gradient, i.e., $\psi_i = \partial\psi/\partial x^i$, and F_3-*planar* if $\psi_i = \varphi_\alpha F_i^\alpha$. If A_n is an equiaffine space, then an F_3-planar mapping is F_2-planar.

The following Theorem holds [7]:

Theorem 1. *An equiaffine space A_n admits F_3-planar mappings onto $\bar{V}_n$ if and only if a regular symmetric tensor a^{ij} and a vector ξ^i satisfy the following equations:*

$$(4) \qquad \text{a)} \quad a^{ij}{}_{,k} = \xi^\alpha F_\alpha^{(i} \delta_k^{j)} + \xi^{(i} F_k^{j)}, \qquad \text{b)} \quad a^{\alpha(i} F_\alpha^{j)} = 0.$$

Solutions of (3) and (4) are connected by the relations

$$a^{ij} = \mathrm{e}^{-2\psi}\bar{g}^{ij}, \quad \xi^i = -\mathrm{e}^{-2\psi}\bar{g}^{i\alpha}\varphi_\alpha,$$

where $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$.

2. Holomorphically projective mappings

An F_3-planar mapping from a space A_n with an affine connection onto a Hermitian space $\bar{H}_n$, for which formulas (2) are satisfied and F_i^h is the almost complex structure of $\bar{H}_n$, is called a *holomorphically projective mapping*.

For this mapping it holds:

$$F^h_{i,j} = F^h_{i|j}, \tag{5}$$

where ‘,’ and ‘ | ’ are the covariant derivatives in A_n and $\bar{H}_n$ respectively.

In the following we will study holomorphically projective mappings from an equiaffine space A_n onto a non-Kählerian space $\bar{H}_n$, in which $F^h_{i|j} \neq 0$.

In this case (5) implies

$$F^h_{i,j} \neq 0. \tag{6}$$

We shall investigate the differential prolongation of the conditions (4b). Let us differentiate them covariantly by x^k in A_n and applying (4a) we find the following:

$$a^{\alpha(i}F^{j)}_{\alpha,k} = 0\,. \tag{7}$$

By further differentiation and simplification we obtain

$$\xi^{(i}F^{j)}_{\alpha}F^{\alpha}_{l,k} - \xi^{\alpha}F^{(i}_{\alpha}F^{j)}_{l,k} + \delta^{(i}_{l}F^{j)}_{\beta}F^{\beta}_{\alpha,k}\xi^{\alpha} - F^{(i}_{l}F^{j)}_{\alpha,k}\xi^{\alpha} = a^{\alpha(i}F^{j)}_{\alpha,kl}\,. \tag{8}$$

Contracting (8) with respect to the indices j and l, we obtain

$$F^i_\beta F^\beta_{\alpha,k}\xi^\alpha = \frac{1}{n+2}(a^{\alpha\beta}F^i_{\alpha,k\beta} + a^{\alpha i}F^\gamma_{\alpha,k\gamma})\,. \tag{9}$$

After contracting (9) with $F^{i'}_i$ we shall replace the index i' by i. We get

$$F^i_{\alpha,k}\xi^\alpha = -\frac{1}{n+2}(a^{\alpha\beta}F^\delta_{\alpha,k\beta} + a^{\alpha\delta}F^\gamma_{\alpha,k\gamma})\,F^i_\delta\,. \tag{10}$$

Substituting (9) and (10) to (8), we find

$$\xi^{(i}F^{j)}_{\alpha}F^{\alpha}_{l,k} - \xi^{\alpha}F^{(i}_{\alpha}F^{j)}_{l,k} = a^{\alpha\beta}\,\overset{1}{T}{}^{(ij)}_{\alpha\beta kl}\,, \tag{11}$$

where

$$\overset{1}{T}{}^{ij}_{\alpha\beta kl} \overset{\text{def}}{=} \delta^i_\alpha F^j_{\beta,kl} - \frac{1}{n+2}\big(F^i_l(F^j_\delta F^\delta_{\alpha,k\beta} + F^\gamma_{\alpha,k\gamma}F^j_\beta) + \delta^i_l(F^j_{\alpha,k\beta} + F^\gamma_{\alpha,k\gamma}\delta^j_\beta)\big).$$

Under the condition (6) we check easily that there exist vectors ε and η such that $a^i \equiv F^i_{k,l}\varepsilon^l\eta^k \not\equiv 0$. Evidently, the vector a^i is not collinear with the vector $a^\alpha F^i_\alpha$. Hence, there exist a covector λ_i such that $\lambda_\alpha a^\alpha = 0$, $\lambda_\alpha F^\alpha_\beta a^\beta = 1$. Contracting (11) with $\varepsilon^l\eta^k$ we obtain

$$\xi^i F^j_\alpha a^\alpha + \xi^j F^i_\alpha a^\alpha - \xi^\alpha F^i_\alpha a^j - \xi^\alpha F^j_\alpha a^i = a^{\alpha\beta} \overset{1}{T}{}^{(ij)}_{\alpha\beta kl}\varepsilon^l\eta^k . \tag{12}$$

Contracting (12) with $\lambda_i\lambda_j$ we check that

$$\xi^\alpha\lambda_\alpha = a^{\alpha\beta} \overset{1}{T}{}^{ij}_{\alpha\beta kl}\varepsilon^l\eta^k\lambda_i\lambda_j$$

and contracting (12) with λ_j we find

$$\xi^i = \varepsilon\, a^i + a^{\alpha\beta} \overset{2}{T}{}^i_{\alpha\beta}, \tag{13}$$

where

$$\varepsilon \overset{\text{def}}{=} \xi^\alpha F^\beta_\alpha\lambda_\beta$$

and

$$\overset{2}{T}{}^i_{\alpha\beta} \overset{\text{def}}{=} (\overset{1}{T}{}^{(i\gamma)}_{\alpha\beta kl} - \overset{1}{T}{}^{(\gamma\delta)}_{\alpha\beta kl}\lambda_\delta F^i_\alpha a^\alpha)\varepsilon^k\eta^l\lambda_\gamma .$$

Applying (13) to (11) we obtain:

$$\varepsilon(a^{(i}F^{j)}_\alpha F^\alpha_{l,k} - a^\alpha F^{(i}_\alpha F^{j)}_{l,k}) = a^{\alpha\beta} \overset{3}{T}{}^{(ij)}_{\alpha\beta kl} , \tag{14}$$

where

$$\overset{3}{T}{}^{ij}_{\alpha\beta kl} \overset{\text{def}}{=} \overset{1}{T}{}^{(ij)}_{\alpha\beta kl} - \overset{2}{T}{}^{(i}_{\alpha\beta} F^{j)}_\gamma F^\gamma_{l,k} + \overset{2}{T}{}^\gamma_{\alpha\beta} F^{(i}_\gamma F^{j)}_{l,k}.$$

The bracket on the left-hand side of (14) must be non-vanishing, otherwise there would be $F^i_{j,k} = 0$, which is in contradiction with (6). So there exists a tensor field Q^{kl}_{ij} satisfying

$$Q^{kl}_{ij}(a^{(i}F^{j)}_\alpha F^\alpha_{l,k} - a^\alpha F^{(i}_\alpha F^{j)}_{l,k}) = 1.$$

Hence from (14) it follows that

$$\varepsilon = a^{\alpha\beta} \overset{3}{T}{}^{ij}_{\alpha\beta kl} Q^{kl}_{ij}$$

and further

$$\xi^i = a^{\alpha\beta}T^i_{\alpha\beta},$$

where

$$T^i_{\alpha\beta} \overset{\text{def}}{=} \overset{2}{T}{}^i_{\alpha\beta} + a^i \overset{3}{T}{}^{\gamma\delta}_{\alpha\beta kl} Q^{kl}_{\gamma\delta}. \tag{15}$$

The following theorem is the result of previous computation and Theorem 1.

Theorem 2. *Let A_n be an equiaffine space with an affine connection, F be a covariantly non-constant almost complex structure (i.e., an affinor F_i^h such that $F_\alpha^h F_i^\alpha = -\delta_i^h$ and $F_{i,j}^h \neq 0$). Then A_n admits a holomorphically projective mapping onto a non-Kählerian Hermitian space $\bar{H}_n$, if and only if the following system of linear differential equations of Cauchy type is solvable with respect to the unknown functions a^{ij}:*

$$a^{ij}{}_{,k} = \xi^\alpha F_\alpha^{(i} \delta_k^{j)} + \xi^{(i} F_k^{j)}, \tag{16}$$

where $\xi^i = a^{\alpha\beta} T_{\alpha\beta}^i$. Further, the matrix (a^{ij}) satisfies $\det\|a^{ij}\| \neq 0$ and the algebraic conditions

$$a^{ij} = a^{ji}, \qquad a^{ij} = a^{\alpha\beta} F_\alpha^i F_\beta^j. \tag{17}$$

Here $T_{\alpha\beta}^i$ is tensor which is explicitly expressed using the objects defined in A_n: (15), *i.e., affine connection A_n and affinor F_i^h.*

This theorem is a generalization of the results in [2], [5]–[9], [15]–[17].

The system (16) does not have more than one solution for the initial Cauchy conditions $a^{ij}(x_o) = a_o^{ij}$ under the conditions (17). Therefore the general solution of (16) does not depend on more than $N_o = n^2/4$ parameters. The question of existence of a solution of (16) leads to the study of integrability conditions, which are linear equations with respect to the unknowns variables a^{ij} with objects from the space A_n.

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