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Skew-symmetric frames and constant curvature[1]

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Abstract. Let (M, g) be a Riemannian manifold and D a distribution in its tangent fiber bundle. We say that D possesses local (not necessarily orthonormal) skew-symmetric frames if around every point of M there are local sections $X_1, \dots, X_k \in \Gamma D$, such that $\nabla_{X_i} X_j + \nabla_{X_j} X_i = 0$, for any $i, j = 1, \dots, k$, where k is the dimension of the distribution, ∇ is the Levi-Civita connection with respect to g and the X_i are linearly independent at every point. It is proven that if a smooth (C^∞) surface possesses (local) skew-symmetric frames (i.e., in the whole tangent distribution), then it has constant curvature.

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Special frames on Riemannian manifolds were studied by many authors, e.g., d'Atri and Nickerson, [1]. In that paper they considered in particular Killing frames, i.e., frames of local Killing vector fields with constant scalar product to each other. They proved (Theorem 3.6, Proposition 3.2) that such a manifold is locally symmetric and all sectional curvatures are non-negative.

We will discuss here some properties of more general skew-symmetric frames. First we recall some basic facts about geodesics and geodesic fields in Riemannian geometry.

Lemma 1. *Let X be a non-vanishing geodesic (auto-parallel) vector field on a Riemannian manifold. Then $X/\|X\|$ is a geodesic vector field, too.*

Proposition 2. *Let (M, g) be a Riemannian manifold and let $p \in M$, $v \in T_pM$. Then there exists locally a geodesic (auto-parallel) vectorfield X on M such that $X_p = v$.*

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

Proposition 3. *Let X_1 be a unit geodesic vectorfield on a surface S. Then there exist local coordinates (u, v) on S such that $X_1 = \partial/\partial v$ and $g(\partial/\partial v, \partial/\partial u) = 0$.*

We may state the main result now.

Theorem 4. *If a smooth (C^∞) surface (locally) admits systems of skew-symmetric frames, then its curvature is constant.*

Proof. The proof is rather computational. Let (u, v) be a local coordinate system of the form given by Proposition 3, so the coefficients of the metric are $g_{11} = E(u, v) > 0$, $g_{12} = g_{21} = 0$, $g_{22} = 1$.

Then the Riemann–Christofell coefficients are given by the formulae

$$\Gamma^1_{11} = \frac{1}{2}\frac{\partial \ln E}{\partial u}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}\frac{\partial \ln E}{\partial v},$$

$$\Gamma^2_{11} = -\frac{1}{2}\frac{\partial E}{\partial v}, \quad \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = \Gamma^2_{21} = 0,$$

while the curvature is given by

$$K = -\frac{1}{\sqrt{E}}\frac{\partial^2}{\partial v^2}(\sqrt{E}).$$

By the hypothesis, local vector fields X, Y are given on S such that $\nabla_X Y + \nabla_Y X = \nabla_X X = \nabla_Y Y = 0$. Denote $X = f_1\, \partial/\partial u + g_1\, \partial/\partial v$ and $Y = f_2\, \partial/\partial u + g_2\, \partial/\partial v$.

Therefore the following equations hold for the skew-symmetric frame X, Y on S:

$$f_1\frac{\partial f_2}{\partial u} + g_1\frac{\partial f_2}{\partial v} + f_2\frac{\partial f_1}{\partial u} + g_2\frac{\partial f_1}{\partial v} + f_1 f_2\frac{\partial \ln E}{\partial u} + (f_1 g_2 + f_2 g_1)\frac{\partial \ln E}{\partial v} = 0,$$

$$f_1\frac{\partial g_2}{\partial u} + g_1\frac{\partial g_2}{\partial v} + f_2\frac{\partial g_1}{\partial u} + g_2\frac{\partial g_1}{\partial v} - f_1 f_2\frac{\partial E}{\partial v} = 0,$$

$$f_i\frac{\partial f_i}{\partial u} + g_i\frac{\partial f_i}{\partial v} + \frac{1}{2} f_i{}^2\frac{\partial \ln E}{\partial u} + (f_i g_i)\frac{\partial \ln E}{\partial v} = 0, \quad i = 1, 2,$$

$$f_i\frac{\partial g_i}{\partial u} + g_i\frac{\partial g_i}{\partial v} - \frac{1}{2} f_i{}^2\frac{\partial E}{\partial v} = 0, \quad i = 1, 2.$$

It follows from the last equations that one can suppose $g_i \neq 0$. Moreover, applying Lemma 1 to the vectorfield X and then Proposition 3 to $X_1 = X/\|X\|$, it follows that one can assume $f_1 = 0$.

It follows that $f_2 = A(u)/E$ and $g_1 = g_1(u) \neq 0$. Then the calculation shows that $E = h^2(v)A^2(u)$ and $f_2 = 1/A(u)h^2(v)$, with functions A, h depending on u (respectively v) only.

The following notation $g_2 = a(u, v)g_1$ is convenient. Compatibility conditions for the system above reduce then to

$$\frac{\partial^2 a}{\partial u \partial v} = \frac{\partial^2 a}{\partial v \partial u}.$$

From the second equation we obtain $\partial a/\partial v = f_2 b'$, while the sixth equation gives

$$\frac{\partial a}{\partial u} = \frac{1}{2} b f_2 \frac{\partial E}{\partial v},$$

where $b = 1/g_1$. Therefore the symmetry of the second derivatives of a gives

$$b'\left(-\frac{A'}{A^2h^2}\right) + b''\frac{1}{Ah^2} = Ab\left(\frac{h'}{h}\right)'.$$

Let us separate the variables (u, v). It follows in particular that $h''h-(h')^2 \equiv \text{const}$, therefore (by Lemma 5) $h''/h \equiv \text{const}$, so finally

$$K = -\frac{1}{\sqrt{E}}\frac{\partial^2}{\partial v^2}(\sqrt{E}\,) \equiv \text{const}\,. \quad \square$$

We have made use of the

Lemma 5. *Let $x \neq 0$ be a real function in one real variable such that $xx'' - (x')^2 \equiv$ const. Then*

$$\frac{x''}{x} \equiv \text{const}\,.$$

Proof. From the equation verified by x one gets the existence of x'''; by differentiating one gets more, $xx''' - x'x'' = 0$. Then $(x''/x)' = (x'''x - x''x')/x^2 = 0$, therefore $x''/x \equiv \text{const}$. $\square$

Remark 6. We remark here that a surface with constant curvature does possess (local) skew-symmetric frames.

We recall here for convenience the construction of (local) skew-symmetric frames on the sphere S^2, (see [2]). First, let us take the field Q on R^3 given in Euclidean coordinates by $Q = zx\,\partial/\partial x + zy\,\partial/\partial y + (z^2 - 1)\,\partial/\partial z$. The straightforward calculation shows that Q restricts to a tangent field Q to S^2 and $\nabla_Q Q = zQ$, where ∇ is the Levi-Civita connection on the sphere. Next, $X = (1/y)Q$ is a geodesic vector field on the hemisphere $y > 0$. In an analogous way let us consider $T = (x^2-1)\,\partial/\partial x + xy\,\partial/\partial y + xz\,\partial/\partial z$ and $Y = (1/y)T$ and the conditions $\nabla_X Y + \nabla_Y X = \nabla_X X = \nabla_Y Y = 0$ and X, Y linearly independent are fulfilled on $y > 0$.

We recall ([3]) the following

Definition 7. A *Riemannian group* is a connected Lie group G furnished with a bi-invariant metric g.

Remark 8. Every Riemannian group (G, g) possesses globally defined skew--symmetric frames. It follows from Theorem 4 that if $\dim G = 2$, then G has constant curvature.

In fact it is known

Proposition 9. *Let* (G, g) *be a Riemannian group such that* $\dim G = 2$*. Then* $K = 0$.

Proof. For a Riemannian group it holds

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z].$$

Let E_1, E_2 be a basis in the Lie algebra $L(G)$ of G such that either

$$\text{a)} \quad [E_1, E_2] = [E_2, E_1] = [E_1, E_1] = [E_2, E_2] = 0,$$

or

$$\text{b)} \quad [E_1, E_2] = -[E_2, E_1] = E_2,\ [E_1, E_1] = [E_2, E_2] = 0.$$

In both cases it follows that

$$R(E_2, E_1)E_2 = -\frac{1}{4}[[E_2, E_1], E_2] = 0.$$

Consequently $K(E_1 \wedge E_2) = 0$. □

Remark 10. Now we can prove that skew-symmetric frames in tangent bundles do exist on spaces with constant sectional curvature. On Hadamard manifolds they can be chosen to be globally defined.

References

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